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On the Regularity of Multivariate Hermite Interpolation[1]

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In this paper a result due to Gevorgian, Sahakian, and the author concerning the regularity of bivariate Hermite interpolation is generalized in two directions: in the bivariate case and for arbitrary dimensions. Also a notion of independence (preregularity) of interpolation conditions is discussed and a relation on the independence on different dimensions is indicated. As corollaries combinatorial inequalities are obtained. At the end a pair of related number inequalities is presented.

1. INTRODUCTION AND PRELIMINARY RESULTS

Hermite interpolation by multivariate algebraic polynomials is studied. The interpolation parameters are the values of a function and its partial derivatives up to some orders $n_\nu - 1$ at given points $x^{(\nu)}$, $\nu = 1, ..., s$, where n_ν is the multiplicity of $x^{(\nu)}$. A point with multiplicity 1 is called simple. The space of interpolating polynomials is

$$\pi_{n,k} := \{\text{polynomials of } k \text{ variables of total degree} \leqslant n\}.$$

We define a scheme

$$\mathcal{N} = \{n_1, ..., n_s; n\}$$

as the sequence of multiplicities, $\mathcal{N}' := \{n_1, ..., n_2\}$, associated with the degree of interpolating polynomials, n. Sometimes, in the scheme, we will also indicate the number of variables of polynomials, k, as a superscript: $\mathcal{N}^k = \{n_1, ..., n_s; n\}^k$.

[1] A portion of the contents of this paper was reviewed in [5].

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Already in the Lagrange multivariate interpolation we deal with a new phenomena—in contrast with the univariate case there are sets of knots for which the interpolation is not correct.

In addition to this, in the case of Hermite multivariate interpolation, there are schemes, called singular (for example $\{2, 2; 2\}^2$), for which the interpolation is never correct, regardless of the chosen sets of knots.

On the other hand, as we will see later, if a scheme is regular, i.e., there is a set of points for which the Hermite interpolation problem is correct, then it is correct for almost all choices of sets. Hence it is important to determine whether a given scheme is regular.

Note that the characterization of regular schemes is the central problem. It is still an open problem even in the extensively studied bivariate case (see [2]).

In this paper the regularity of schemes of a substantial class of multivariate interpolation problems will be established.

One should mention that the problem of regularity of schemes was considered first in algebraic geometry by Nagata [8]. In approximation theory the investigation was initiated by G. G. Lorentz and R. A. Lorentz [6].

For $x := (x_0, ..., x_{k-1})$ and the multi-index $\alpha = (\alpha_0, ..., \alpha_{k-1}) \in Z^k_+$ we denote

$$x^\alpha := x_0^{\alpha_0} \cdots x_{k-1}^{\alpha_{k-1}},$$

$$D^\alpha P := \frac{\partial^{|\alpha|} P}{\partial x_0^{\alpha_0} \cdots \partial x_{k-1}^{\alpha_{k-1}}}, \qquad |\alpha| := \alpha_0 + \cdots + \alpha_{k-1}.$$

DEFINITION 1.1. For the scheme $\mathcal{N} = \{n_1, ..., n_s; n\}$ and the point set $\mathcal{X} = \{x^{(1)}, ..., x^{(s)}\} \subset R^k$ the Hermite interpolation problem $(\mathcal{N}, \mathcal{X}) = (\mathcal{N}, \mathcal{X})^k$ is correct if we can find a unique polynomial $P \in \pi_{n,k}$ satisfying

$$D^\alpha P(x^{(\nu)}) = \lambda_{\alpha,\nu}, \qquad |\alpha| \leqslant n_\nu - 1, \quad \nu = 1, ..., s, \tag{1.1}$$

for any given data

$$\Lambda = \{\lambda_{\alpha,\nu} : |\alpha| \leqslant n_\nu - 1, \nu = 1, ..., s\}.$$

In what follows, we briefly express equalities of form (1.1) by writing: $D^{\mathcal{N}} P|_{\mathcal{X}} = \Lambda$. Denote

$$\mathcal{N}_\# := \mathcal{N}^k_\# := \sum_{\nu=1}^{s} (n_\nu - 1)^{(k)} \qquad \text{and} \qquad d_{n,k} := \dim \pi_{n,k} = n^{(k)},$$

where $m^{(k)} := \binom{m+k}{k}$, $(-1)^{(k)} = 0$.

Note that $\mathcal{N}_{\#}$ is the number of conditions in (1.1) and $d_{n,k}$ is the number of coefficients of polynomials from $\pi_{n,k}$. Now observe that (1.1) can be considered as a system of $\mathcal{N}_{\#}$ linear equations with respect to $d_{n,k}$ unknown coefficients of $P \in \pi_{n,k}$. This is the key in establishing of the forthcoming assertions (see [2, 3]).

Denote by $[\mathcal{X}]_{\mathcal{N}} = [\mathcal{X}]^k_{\mathcal{N}}$ the $(\mathcal{N}_{\#} \times d_{n,k})$ matrix of system (1.1), which consists of the rows

$$D^\alpha\{x^\beta: |\beta| \leqslant n\}|_{x^{(\nu)}} = D^\alpha\{1, x_0, ..., x_{k-1}, ..., x_0^n, ..., x_{k-1}^n\}|_{x^{(\nu)}}, \quad |\alpha| \leqslant n_\nu - 1, \qquad \nu = 1, ..., s. \tag{1.2}$$

Assertion 1.2. The condition

$$\mathcal{N}^k_{\#} = d_{n,k}$$

is necessary for the correctness of the problem $(\mathcal{N}, \mathcal{X})^k$.

We call $\mathcal{N}$ an exact scheme (in k dimension) if the above equality holds.

Assertion 1.3. If the problem $(\mathcal{N}, \mathcal{X})$ is correct for some $\mathcal{X}$, then it is correct for almost all $\mathcal{X} \in R^{ks}$ (with respect to the Lebesgue measure in R^{ks}).

More precisely, it is correct if and only if $\mathcal{X}$ does not belong to a hypersurface in R^{ks}. Namely, this is the hypersurface determined by the equation $\det[\mathcal{X}]_{\mathcal{N}} = 0$. The determinant here is a (ks)-variate polynomial (not identically zero) with respect to the coordinates of $x^{(i)}$, $i = 1, ..., s$.

Assertion 1.4. For any exact scheme $\mathcal{N}$ the problem $(\mathcal{N}, \mathcal{X})^k$ is correct if and only if

$$P \in \pi_{n,k}, \quad D^{\mathcal{N}} P|_{\mathcal{X}} = 0 \qquad \text{imply} \quad P = 0.$$

Definition 1.5. We say that the exact scheme $\mathcal{N}$ is

(i) regular if the problem $(\mathcal{N}, \mathcal{X})$ is correct for some $\mathcal{X}$;

(ii) singular if $(\mathcal{N}, \mathcal{X})$ is not correct regardless of the choice $\mathcal{X}$.

In other words, the exact scheme $\mathcal{N}$ is singular if $\det[\mathcal{X}]_{\mathcal{N}}$ vanishes identically and is regular otherwise.

Definition 1.6. The interpolation problem $(\mathcal{N}, \mathcal{X})$ is called

(i) independent (dependent) if the corresponding interpolation conditions also are, i.e., the $\mathcal{N}_{\#}$ rows of $[\mathcal{X}]_{\mathcal{N}}$ are independent (dependent);

(ii) solvable if for any given data Λ there is a (not necessarily unique) polynomial $P \in \pi_{n,k}$ satisfying $D^{\mathcal{N}} P|_{\mathcal{X}} = \Lambda$.

Let us denote

$$\mathcal{N}_{-i} := \{(n_1 - i)_+, ..., (n_s - i)_+; (n-i)_+\}, \qquad i \geqslant 0.$$

REMARK 1.7. (i) *The problem* $(\mathcal{N}, \mathcal{X})$ *is independent if and only if it is solvable.*

(ii) *If the problem* $(\mathcal{N}, \mathcal{X})$ *is independent, then so is the problem* $(\mathcal{N}_{-1}, \mathcal{X})$.

(iii) *The problem* $(\mathcal{N}, \mathcal{X})$ *is independent if and only if the problem* $(\mathcal{N}, T\mathcal{X})$ *is independent, where* T *is a nonsingular linear transformation.*

Indeed, we will get (i) by using a standard argument on matrices: The rows of a matrix are independent if and only if the columns are complete, i.e., any vector of appropriate dimension can be presented as linear combination of columns. Next, if $P \in \pi_{n,k}$ and

$$D^\alpha P(x^{(\nu)}) = \lambda_{\alpha,\nu}, \qquad |\alpha| \leqslant n_\nu - 1, \quad \nu = 1, ..., s,$$

then $Q := \partial P / \partial x_0 \in \pi_{n-1,k}$ and

$$D^\alpha Q(x^{(\nu)}) = \lambda_{\alpha + e_0, \nu}, \qquad |\alpha| \leqslant n_\nu - 2, \quad \nu = 1, ..., s,$$

with $e_0 = (1, 0, ..., 0)$. Thus if $(\mathcal{N}, \mathcal{X})$ is solvable, then so is the problem $(\mathcal{N}_{-1}, \mathcal{X})$.

To prove (iii) note that, for the polynomial $R(x) = P(Tx)$, each coordinate of the vector $\vec{a} = \{D^\alpha R(x^{(0)}) : |\alpha| = m\}$ is a linear combination of coordinates of $\vec{b} = \{D^\beta P(y^{(0)}) : |\beta| = m\}$, where $y^{(0)} = T(x^{(0)})$. Hence there is a matrix $\mathcal{A}$ of appropriate dimensions such that $\vec{a} = \mathcal{A}\vec{b}$. Moreover, the matrix $\mathcal{A}$ is not singular, since in the same way $\vec{b}$ can be represented via $\vec{a}$. It remains to note that

$$P(x) \in \pi_{n,k} \Leftrightarrow P(Tx) \in \pi_{n,k}$$

and to make use of part (i).

The independence of the problem $(\mathcal{N}, \mathcal{X})$ is equivalent also to the condition that some minor of $[X]_{\mathcal{N}}$ of order $\mathcal{N}_\#$ is not zero. Therefore we have, similarly to Assertion 1.3, that the independence of the problem $(\mathcal{N}, \mathcal{X})$ for some $\mathcal{X}$ implies its independence for almost all $\mathcal{X} \in R^{ks}$.

DEFINITION 1.8. We say that the scheme $\mathcal{N}$ is

(i) independent, if the problem $(\mathcal{N}, \mathcal{X})$ for some $\mathcal{X}$ also is;

(ii) dependent, if the problem $(\mathcal{N}, \mathcal{X})$ for all $\mathcal{X}$ also is.

It is worth mentioning that the regularity (independence) of $\mathcal{N}$ remains unchanged in the complex interpolation case, i.e., $(\mathcal{N}, \mathcal{X})$ is correct (independent) for some $\mathcal{X} \in R^{ks}$ if and only if it is correct (independent) for some $\mathcal{X} \in C^{ks}$. (In the latter case the coefficients of interpolating polynomials and the data are complex, too.)

Indeed, both cases are equivalent to the condition that a polynomial of ks variables, namely $\det[\mathcal{X}]_{\mathcal{N}}$ (or some its minor), does not vanish identically. It remains to note that $p(x)$ is (not) identically zero if and only if $p(z)$ is (not) identically zero, where p is a polynomial of m variables, $x \in R^m$, $z \in C^m$.

We get immediately from Remark 1.7(ii):

COROLLARY 1.9. *If the scheme $\mathcal{N}$ is independent, then so is $\mathcal{N}_{-1}$.*

Now we are in a position to prove the following relation concerning the independence of the problem $(\mathcal{N}, \mathcal{X})^k$ for various k. Here we are identifying R^k with the subspace

$$L: \{x \in R^m : x_k = 0, ..., x_{m-1} = 0\},$$

THEOREM 1.10. *The problem $(\mathcal{N}, \mathcal{X})^m$ will be independent, with all the points of $\mathcal{X}$ belonging to the k-dimensional $(1 \leqslant k < m)$ linear subspace $L \subset R^m$ if and only if the problem $(\mathcal{N}, \mathcal{X})^k$ is independent.*

Proof. It is enough to consider only the case $k = m-1$. Set $q := \max\{n_1, ..., n_s\} - 1$. We have that $q \leqslant n$ (the equality here holds for the Taylor case $\mathcal{N} = \{n+1; n\}$ only). Now observe that after certain interchanges between the rows and columns, the matrix $[\mathcal{X}]^m_{\mathcal{N}}$ can be reduced to the matrix

$$\begin{bmatrix} \mathcal{A}_0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_0 & 0 & \cdots & 0 \end{bmatrix},$$

where

$$\mathcal{A}_i = i!\,[\mathcal{X}]^{(m-1)}_{\mathcal{N}_{-i}},$$

and 0 means a zero matrix of an appropriate dimension. Note that $\mathcal{A}_i$ includes those elements of $[\mathcal{X}]^m_{\mathcal{N}}$ which are in the columns corresponding to x^β, with $\beta_{m-1} = i$ (see (1.2)) and in the rows corresponding to D^α, with $\alpha_{m-1} = i$, while the 0-columns above correspond to x^β with $\beta_{m-1} > q$.

This, in view of Remark 1.7(ii), ends the proof.

It follows from the proof that in general the number of independent rows of the matrix $[\mathscr{X}]^m_{\mathscr{N}}$, with sets $\mathscr{X}$ as in Theorem 1.10, equals the sum of similar numbers of the matrices $\mathscr{A}_i$, $i=0, ..., q$.

Note also that in view of Remark 1.7(iii), Theorem 1.10 is valid in the case of any m-dimensional linear subspace L.

Corollary 1.11. *If the scheme $\mathscr{N}^k$ is independent, then so is the scheme $\mathscr{N}^m$ for any $m \geqslant k$.*

It is easily seen that if the problem $(\mathscr{N}, \mathscr{X})^k$ is independent then

$$\mathscr{N}^k_{\#} \leqslant d_{n,k}; \tag{1.3}$$

in other words, the number of independent rows of the matrix $[\mathscr{X}]_{\mathscr{N}}$ is less than or equal to the number of columns.

We associate a scheme $\mathscr{N} = \{n_1, ..., n_s; n\}$ that satisfies the above inequality with the exact scheme

$$\bar{\mathscr{N}} = \{n_1, ..., n_s, n_{s+1}, ..., n_r; n\},$$

with $n_{s+1} = \cdots = n_r = 1$.

Accordingly, for $\mathscr{X} = \{x^{(1)}, ..., x^{(s)}\}$ we define

$$\bar{\mathscr{X}} := \{x^{(1)}, ..., x^{(s)}, x^{(s+1)}, ..., x^{(r)}\},$$

where $x^{(s+1)}, ..., x^{(r)}$ are some (simple) points.

The next assertion allows us to interpret the independence property as a "preregularity" property.

Assertion 1.12. (i) The problem $(\bar{\mathscr{N}}, \bar{\mathscr{X}})$ is correct, for some $\bar{\mathscr{X}}$, if and only if the problem $(\mathscr{N}, \mathscr{X})$ is independent.

(ii) The scheme $\bar{\mathscr{N}}$ is regular if and only if $\mathscr{N}$ is independent.

Actually, a proof of (i) can be found in [7, Chapter 1, Lemma 2.1], while (ii) follows immediately from (i).

By changing slightly the arguments used in the above reference one can prove a stronger result:

Remark 1.13. *Let the (Hermite) interpolation problem $(\mathscr{N}, \mathscr{X})$ be independent and let $\mathscr{L}$ be any set of simple points correct for the Lagrange interpolation problem (both problems considered for $\pi_{n,k}$). Then the process of successive addition of simple points to $\mathscr{X}$ from $\mathscr{L}$, by checking the independence of interpolation conditions of resulting sets, necessarily leads to a set $\bar{\mathscr{X}}$ such that the Hermite interpolation problem $(\bar{\mathscr{N}}, \bar{\mathscr{X}})$ is correct.*

Proof. We need to check only the following induction step: There is a point x in $\mathscr{L}$ for which the row

$$(1, x_0, ..., x_{k-1}, ..., x_0^n, ..., n_{k-1}^n)$$

is linearly independent of $r := \mathscr{N}_{\#}$ rows of matrix

$$[\mathscr{X}]_{\mathscr{N}} =: \mathscr{A} = (a_{i,\beta}) \qquad (i = 1, ..., r, |\beta| \leqslant n).$$

To this end we fix r linearly independent columns $\beta_1, ..., \beta_r$ and any another column β_{r+1} in $\mathscr{A}$. Then we consider the polynomial

$$D(x) = \det \begin{bmatrix} a_{1,\beta_1} & \cdots & a_{1,\beta_r} & a_{1,\beta_{r+1}} \\ \cdots & & \cdots & \cdots \\ a_{r,\beta_1} & \cdots & a_{r,\beta_r} & a_{r,\beta_{r+1}} \\ x^{\beta_1} & \cdots & x^{\beta_2} & x^{\beta_{r+1}} \end{bmatrix}.$$

This is a nonzero polynomial (the coefficient of $x^{\beta_{r+1}}$ does not vanish) from $\pi_{n,k}$. Hence it takes a nonzero value at some point x of $\mathscr{L}$. This ends the proof.

Let $\mathscr{X} = \{x^{(v)}\}_{v=1}^s$ and $\tilde{\mathscr{X}} = \{\tilde{x}^{(v)}\}_{v=1}^s$.

REMARK 1.14. (i) *If the problem $(\mathscr{N}, \mathscr{X})$ is independent (for $\pi_{n,k}$), then so is the problem $(\mathscr{N}, \tilde{\mathscr{X}})$, with $\tilde{x}^{(v)}$ sufficiently close to $x^{(v)}$, $v = 1, ..., s$.*

(ii) *If the problem $(\mathscr{N}, \mathscr{X})^k$ is independent, with $x^{(v)}$ all belonging to a k-dimensional $(1 \leqslant k < m)$ linear subspace L, then so is the problem $(\mathscr{N}, \tilde{\mathscr{X}})^m$, where the projection of $\tilde{x}^{(v)}$ on L is $x^{(v)}$, $v = 1, ..., s$.*

Indeed, as was mentioned earlier, the independence of the problem $(\mathscr{N}, \mathscr{X})$ is equivalent to the condition that some minor of $[X]_{\mathscr{N}}$ is not zero. This implies that the sets of points, at which interpolation conditions are independent, is open or empty. Thus we get (i).

To prove (ii), assume without loss of generality (Remark 1.7(iii)), that $L = \{x \in R^m : x_k = 0, ..., x_{m-1} = 0\}$. Then we have $x_i^{(v)} = \tilde{x}_i^{(v)}$ for $i = 0, ..., k-1$ and $x_i^{(v)} = 0$, for $i = k, ..., m-1$; $v = 1, ..., s$. Now consider the linear transformation

$$T: (x_0, ..., x_{m-1}) \to (x_0, ..., x_{k-1}, \varepsilon x_k, ..., \varepsilon x_{m-1}).$$

It is easily seen that the image of $\tilde{X}$, i.e. $T(\tilde{\mathscr{X}})$, gets arbitrarily close to $\mathscr{X}$ as ε tends to 0. To end it remains to make use of the point (i) and, once more, Remark 1.7(iii).

COROLLARY 1.15 (Severi [9]). *The problem* $(\mathcal{N}, \mathcal{X})^k$ *is independent for all* $\mathcal{X}$ *if and only if* $n_1 + \cdots + n_s \leqslant n + 1$.

Indeed, we have from Theorem 1.10 that the problem $(\mathcal{N}, \mathcal{X})^1$ is independent for any $\mathcal{X}$, with all its points belonging to a 1-dimensional linear subspace L, if and only if the problem $(\mathcal{N}, \mathcal{X})^k$ is independent for any such $\mathcal{X}$.

On the other hand, on account of Remark 1.14(ii), the independence of the problem $(\mathcal{N}, \mathcal{X})^k$ for any above $\mathcal{X}$ implies its independence for arbitrary $\mathcal{X}$. Meanwhile, if necessary, in view of Remark 1.7(iii), we can use a linear transformation to make the projections of the points on L different. Now it remains to use the well-known univariate result.

2. THE MAIN RESULT

The result from [1, Theorem 3.3] that we generalize can be formulated in the following way:

THEOREM 2.1 (Gevorgian, Hakopian, Sahakian). *Let* $\mathcal{N} = \{n_1, ..., n_s; n\}$ *be a scheme with non-increasing sequence of multiplicities satisfying the conditions*

$$n_1 + n_2 + n_3 \leqslant n, \qquad \sum_{i=1}^{s} n_i \leqslant 3n. \tag{2.1}$$

Then the problem $(\mathcal{N}, \mathcal{Y})^2$ *is independent with any set*

$$\mathcal{Y} = \{y_i\}_{i=1}^{s} \subset \{(x, x^3) \colon x > 0\}.$$

Note that this theorem, in turn, is an extension of a result due to Nagata [8], where one has only the existence of the set $\mathcal{Y}$ and the condition $s \leqslant 9$ instead of the second condition of (2.1). We generalize Theorem 2.1 in two directions: in the bivariate case and for arbitrary dimensions. The result (announced in [4] and reviewed in [5]) is a scale of type (2.1) conditions implying the above independence for the general multivariate case:

THEOREM 2.2. *Let* $\mathcal{N} = \{n_1, ..., n_s; n\}$ *be a scheme with non-increasing sequence of multiplicities satisfying for some* q, $1 \leqslant q \leqslant s$, *the conditions*

$$\sum_{i=1}^{q} n_i \leqslant A, \qquad \sum_{i=1}^{s} n_i \leqslant q^{k-1} A, \tag{2.2}$$

where

$$A := A(n, q) := n + 1 - \frac{(q-1)(k-1)}{2}.$$

Then the problem $(\mathcal{N}, \mathcal{Y})^k$ *is independent with any set*

$$\mathcal{Y} = \{y_i\}_{i=1}^s \subset \{(x, x^q, ..., x^{q^{k-1}}): x > 0\}.$$

Note that the case $q=1$ (according to the proof the above condition $x>0$ can be omitted here) in view of Remark 1.14(ii) is equivalent to the result of Severi. Next, in the case $k=2$, $q=3$ we clearly have Theorem 2.1. Note also that in the case $k=2$, $q=2$, using reduction (see [2]), one can replace the condition $n_1+n_2 \leqslant n+1/2$ by a slightly weaker one: $n_1+n_2 \leqslant n+1$.

Theorem 2.2 combined with Assertion 1.12(ii) implies

COROLLARY 2.3. *Let* $\mathcal{N} = \{n_1, ..., n_s; n\}$ *be as in Theorem* 2.2. *Then* $\bar{\mathcal{N}}$ *is regular.*

Now, we will outline briefly how Theorem 2.2 will be proved. Let us start by fixing the set of points $\mathcal{Y}$ with

$$y_i = (x_{0i}, x_{0i}^q, ..., x_{0i}^{q^{k-1}}), \qquad x_{0i} > 0, \quad i = 1, ..., s.$$

Denote also $\mathcal{Y}^* := \{y_i^*\}_{i=1}^s$, where

$$y_i^* = (x_{0i}, x_{0i}^q, ..., x_{0i}^{q^{l-1}})$$

and l ($l = 1, ..., k$) is the number of variables which will be clear from the context.

To prove Theorem 2.2, in view of Assertion 1.12(i) it is enough to show that one can add a set of simple points $\mathcal{X}$ to $\mathcal{Y}$, so that the number of interpolation conditions imposed by the resulting set is equal to $\dim \pi_{n,k}$ and the resulting Hermite interpolation problem is regular.

The latter, in view of Assertion 1.4, is equivalent to the following: a polynomial from $\pi_{n,k}$, for which the above interpolation data is zero, is identically zero.

Actually, the interpolation conditions imposed by the sequence of multiplicities $\mathcal{N}'$ at the points of the set $\mathcal{Y}^*$ are independent also for certain spaces $\pi_{n,l,q}$ of l-variate polynomials which will be defined later. Namely, we have

PROPOSITION 2.4. *Let* $1 \leqslant l \leqslant k$ *and let* $\mathcal{N} = \{n_1, ..., n_s; n\}$ *be as in Theorem* 2.2. *Then there is a set of points* $\mathcal{X} = \mathcal{X}(n, l, q) = \{x^{(i)}\}_{i=1}^{m(l)}$ *with* $m(l) = m(n, l, q) \geqslant 0$ *determined by the equality*

$$\sum_{i=1}^{s} \binom{n_i + l - 1}{l} + m(l) = \dim \pi_{n, l, q}, \tag{2.3}$$

such that the conditions

$$\begin{aligned} D^\alpha p(y_i^*) &= 0, \qquad |\alpha| \leqslant n_i - 1, \quad i = 1, ..., s, \\ p(x^{(i)}) &= 0, \qquad i = 1, ..., m(l), \end{aligned} \tag{2.4}$$

imply that the polynomial $p \in \pi_{n, l, q}$ *vanishes identically.*

Denote

$$\alpha * q := \sum_{i=0}^{k-1} \alpha_i q^i \qquad \text{for} \quad k\text{-multi-index } \alpha = (\alpha_0, ..., \alpha_{k-1}).$$

Now let us introduce the above-mentioned space of l-variate polynomials

$$\pi_{n, l, q} = \left\{ \sum_{|\alpha| + |\beta| \leqslant n} a_{\alpha, \beta} \bar{x}^\alpha x_{l-1}^{\beta * q} : a_{\alpha, \beta} \text{ is real} \right\},$$

where $\bar{x} = (x_0, ..., x_{l-2})$ and α, β are $l-1$ and $k - l + 1$-multi-indices respectively. It is easily seen that the polynomial space $\pi_{n, k, q}$ coincides with $\pi_{n, k}$. Therefore, in the case $l = k$ Proposition 2.4 coincides with Theorem 2.2.

Thus we need only prove Proposition 2.4. This is carried out in several steps.

Proof. **Step 0.** We establish the following property of a scheme $\mathcal{N} = \{n_1, ..., n_s; n\}$, under the assumption that the sequence of multiplicities, $\mathcal{N}'$, is non-increasing, and satisfies (2.2). Namely we will prove that in this case the sequence $\mathcal{N}'_{-v}$ satisfies (2.2)

(i) with $A(n - vq, q)$ for $v = 1, ..., n_q$,

and

(ii) with $A(n - n_q q + n_q - v, 1)$ for $v = n_q + 1, ..., n - n_q q + n_q$.

To check (i) we show that, if $\mathcal{N}$ satisfies (2.2) and $n_q \neq 0$, then the sequence $\mathcal{N}'_{-1}$ satisfies (2.2) with $A(n - q, q)$. Indeed, in this case we have

$$\sum_{i=1}^{q} (n_i - 1)_+ = \sum_{i=1}^{q} (n_i - 1) = \sum_{i=1}^{q} n_i - q \leqslant A - q = A(n - q, q).$$

Therefore the first inequality of (2.2) is satisfied.

To check the second inequality of (2.2), suppose $n_s \neq 0$, and add a number of $1-s$ to its left-hand side in order to come to an equality $\sum_{i=1}^{s'} n_i = q^{k-1}A$, where $n_i = 1$ for $i > s$. Now, the first inequality of (2.2) implies that $s' \geqslant q^k$, and we get

$$\sum_{i=1}^{s} (n_i - 1)_+ = \sum_{i=1}^{s'} (n_i - 1) \leqslant q^{k-1}A - q^k = q^{k-1}A(n-q, q).$$

To check (ii) suppose that $n_q = 0$. Since the sequence of multiplicities of $\mathcal{N}$ is non-increasing and it satisfies the first inequality of (2.2), we obtain the inequality $\sum_{i=1}^{s} n_i \leqslant n+1$, which is equivalent to the both inequalities of (2.2) for $q=1$. Now, it remains to apply case (i) for $q=1$.

Step 1. We prove the case $l=1$ in Proposition 2.4. To this end consider the class of univariate polynomials

$$\pi_{n,1,q} = \left\{ \sum_{|\alpha| \leqslant n} a_\alpha x_0^{\alpha * q} : a_\alpha \text{ is real} \right\},$$

where α is k-multi-index. This class is connected with the class $\pi_{n,k}$ in the following way:

$$\text{if} \quad p(x) = \sum_{|\alpha| \leqslant n} a_\alpha x^\alpha \in \pi_{n,k}$$

$$\text{then} \quad p_0(x_0) := p(x_0, x_0^q, ..., x_0^{q^{k-1}}) \in \pi_{n,1,q}.$$

To start the proof of the case $l=1$ let us denote

$$M := M(n, 1, q) := \dim \pi_{n,1,q}.$$

Descartes' sign change rule implies that a polynomial $p \in \pi_{n,1,q}$ equals to zero identically, if the number of its positive roots equals to M. (Note that in the case $q=1$, since $\pi_{n,1,1} = \pi_{n,1}$, we can omit here the word "positive" —cf. the remark after Theorem 2.2). Therefore we need only prove that

$$\sum_{i=1}^{s} n_i \leqslant M. \tag{2.5}$$

First we shall obtain an explicit representation for M.

To this end we slightly modify the above definition of the class $\pi_{n,1,q}$ (without changing the class itself), by imposing some additional conditions on the multi-index α under the sign $\sum$ there, which will make different monomials in the sum to correspond to different multi-indices α.

It is not hard to check that the conditions $\{\alpha_i \leqslant q-1, i=0, ..., k-2\}$ are appropriate for this.

Therefore we get

$$M(n, 1, q) = \# I(n, 1, q), \tag{2.6}$$

where

$$I(n, 1, q) := \{\alpha = (\alpha_0, ..., \alpha_{k-1}) \colon |\alpha| \leqslant n, \alpha_i \leqslant q-1, i=0, ..., k-2\}.$$

Now let us prove the inequality

$$M(n, 1, q) \geqslant q^{k-1} A(n, q),$$

which implies (2.5) in view of (2.2).

We have from (2.6) that

$$\begin{aligned} M &= \sum_{\alpha_0=0}^{q-1} \cdots \sum_{\alpha_{k-2}=0}^{q-1} (n+1-\alpha_0 - \cdots - \alpha_{k-2})_+ \\ &\geqslant \sum_{\alpha_0=0}^{q-1} \cdots \sum_{\alpha_{k-2}=0}^{q-1} (n+1-\alpha_0 - \cdots - \alpha_{k-2}) \\ &= (n+1)\, q^{k-1} - (k-1)\, q^{k-2} \sum_{\alpha_0=0}^{q-1} \alpha_0 \\ &= q^{k-2} \left[(n+1)\, q - \frac{(k-1)(q-1)\, q}{2} \right] = q^{k-1} A. \end{aligned}$$

Note that here the equality is attained if and only if $(k-1)(q-1) \leqslant n+1$.

In Steps 2 and 3 we will prove Proposition 2.4 by induction on l. The case $l=1$ was proved in Step 1. Now assume the validity for $l-1$; we will prove it for l.

Step 2. In this step we carry out a procedure described below. Suppose that the conditions (2.4) hold for a polynomial $p \in \pi_{n, l, q}$. First note that

$$p(\bar{x}, x_{l-1}) \in \pi_{n, l, q} \qquad \text{implies} \quad p(\bar{x}, x_{l-2}^q) \in \pi_{n, l-1, q}.$$

Then, by using induction hypothesis and imposing conditions at certain simple points of hypersurface $x_{l-1} = x_{l-2}^q$, we make the polynomial

$p(\bar{x}, x_{l-1}) \in \pi_{n,l,q}$ vanish on this hypersurface. This, in view of Bézout's theorem, implies the factorization

$$p(\bar{x}, x_{l-1}) = (x_{l-1} - x_{l-2}^q)\, p_1(\bar{x}, x_{l-1}). \tag{2.7}$$

Note that the above and following simple points (slightly shifted) later constitute the set $\mathscr{X}$ of Proposition 2.4. Next observe that

$$p \in \pi_{n,l,q} \text{ and } p(\bar{x}, x_{l-1}) = (x_{l-1} - x_{l-2}^q)\, p_1(\bar{x}, x_{l-1}) \text{ imply } p_1 \in \pi_{n-q,l,q}.$$

On the other hand, in view of (2.7), the sequence of multiplicities of p_1 at the set $\mathscr{Y}^*$ coincides with $\mathscr{N}'_{-1}$. This sequence, according to Step 0, satisfies the inequalities (2.2) with $A(n-q, q)$, provided $n_q \neq 0$. Therefore using again the induction hypothesis and imposing conditions at some simple points we can make p_1 vanish on the hypersurface $x_{l-1} = q_{l-2}^q$. Then one can factorize p_1 and similarly the subsequent factors

$$p_j(\bar{x}, x_{l-1}) = (x_{l-1} - x_{l-2}^q)\, p_{j+1}(\bar{x}, x_{l-1}), \tag{2.8}$$

where $p_i \in \pi_{n-iq,l,q}$. Note that we use

$$m(n-jq, l-1, q) = M(n-jq, l-1, q) - \sum_{i=1}^{s} \binom{(n_i - j)_+ + l - 2}{l-1}$$

simple points to get the factorizations (2.7) ($j=0$) and (2.8).

In the case of $q \neq 1$ we finish the procedure with the factorization (2.8) with $j_1 = n_q - 1$, i.e., when 1 sits in the qth place of the sequence $\mathscr{N}'_{-j_1}$ of multiplicities of p_{j_1}. Now let us prove that $n - n_q q \geqslant 0$, which means that the described procedure is correct, i.e., we deal, in the factorizations there, with polynomial spaces of nonnegative degrees. Indeed, according to Step 0, the sequence of multiplicities of the last factor p_{n_q}, i.e., $\mathscr{N}'_{-n_q}$, satisfies the inequality (2.2) with $A(n - n_q q, q)$. Hence

$$0 \leqslant \sum_{i=1}^{q} (n_i - n_q)_+ \leqslant A(n - n_q q, q) \leqslant n - n_q q + 1. \tag{2.9}$$

It remains to note that in the case $q \neq 1$ the last inequality of (2.9) is strict. In the case $q = 1$ we continue the factorizations until the stage $j_2 = n$, i.e., until the degree of the polynomial space containing the last factor p_{j_2+1} equals -1 and consequently $p_{j_2+1} = 0$. Note that the inequality $n_1 \leqslant n+1$ (cf. (2.9) with $q=1$) implies that the sequence of multiplicities of p_{j_2+1}: $\mathscr{N}'_{-j_2-1}$ consists of zeros only.

Step 3. Here we will show that by slightly perturbing the simple points added in Step 2 we obtain a set $\mathscr{X}$ satisfying Proposition 2.4.

The next lemma is the basic tool we shall use in this step.

LEMMA 2.5. *Let* $\mathcal{X}(n-(j+1)\,q, l, q)$ *and* $\mathcal{X}(n-jq, l-1, q)$ *be some sets satisfying Proposition* 2.4. *Then there is a set* $\mathcal{X}'(n-(j+1)\,q, l, q)$ *of points sufficiently close to those of the first set, such that*

$$\mathcal{X}(n-jq, l, q)=\mathcal{X}'(n-(j+1)\,q, l, q)\cup\mathcal{X}(n-jq, l-1, q);$$

i.e., the set in the right-hand side satisfies Proposition 2.4 *for the triplet of parameters* $(n-jq, l, q)$.

Proof. First we slightly change positions of the points of the set $\mathcal{X}(n-(j+1)\,q, l, q)$ in order to get a set $\mathcal{X}'(n-(j+1)\,q, l, q)$ which has no points on the hypersurface $x_{l-1}=x_{l-2}^{q}$. Now, in view of factorization (2.8) and Remark 1.14(i), we need only prove the equality

$$m(n-jq, l, q)=m(n-(j+1)\,q, l, q)+m(n-jq, l-1, q).$$

This, in view of (2.3) and the identity

$$\binom{v+1}{l}=\binom{v}{l}+\binom{v}{l-1},$$

follows from the equality

$$M(n-jq, l, q)=M(n-(j+1)\,q, l, q)+M(n-jq, l-1, q).$$

To check this we first get the explicit representation

$$M(n-jq, l, q)=\#I,$$

where

$$\begin{aligned} I&=I(n-jq, l, q)\\ &=\{\gamma=(\alpha, \beta)=(\alpha_0, ..., \alpha_{l-2}, \beta_0, ..., \beta_{k-l}):\\ &\qquad |\gamma|\leqslant n-jq, \beta_i\leqslant q-1, i=0, ..., k-l-1\}. \end{aligned}$$

This follows by the same method as the representation (2.6) in Step 1.

We conclude from this that

$$\begin{aligned} \#I&=\#\{I(n-jq, l, q): \alpha_{k-2}\geqslant q\}+\#\{I(n-jq, l, q): \alpha_{k-2}\leqslant q-1\}\\ &=M(n-(j+1)\,q, l, q)+M(n-jq, l-1, q). \end{aligned}$$

This completes the proof of Lemma 2.5.

We complete now the proof of Proposition 2.4. Let us first check the case $q=1$ (without appealing to Severi's result). In view of Lemma 2.5 we need only show that there is as set $\mathscr{X}(-1, l, 1)$. The latter is obvious since for the considered parameters (n, l, q) the sequence $\mathscr{N}$ satisfying (2.2) turns into a null sequence and the polynomial class consists of zero only.

Next, to prove the general case, we need to show that there is a set $\mathscr{X}(n-n_q q, l, q)$ satisfying Proposition 2.4. To this end, note that according to the case $q=1$, for any set $\mathscr{Y}^*$ of points on the line $x_1 = \cdots = x_l$ and the space of polynomials

$$\pi_{n-n_q q,\, l} = \pi_{n-n_q q,\, l,\, 1}$$

there is a set $\mathscr{X}(n-n_q q, l, 1)$. Next, on account of Remark 1.14(ii) (cf. also the proof of Corollary 1.15), we can assume that the set of points coincides with $\mathscr{Y}^*$. Now, to end the proof it remains to note only that

$$\pi_{n-n_q q,\, l,\, 1} \subset \pi_{n-n_q q,\, l,\, q}$$

and therefore the set $\mathscr{X}(n-n_q q, l, 1)$ can be extended up to a set $\mathscr{X}(n-n_q q, l, q)$ satisfying Proposition 2.4 (cf. Assertion 1.12(i)).

3. SOME INEQUALITIES

In view of Corollaries 1.9 and 1.11 we get from (1.3)

Corollary 3.1. *Let* $\mathscr{N} = \{n_1, ..., n_s : n\}^k$ *be an independent scheme; then*

$$\sum_{i=1}^{s} \binom{(n_i - l)_+ + m - 1}{m} \leqslant \binom{(n-l)_+ + m}{m}, \qquad \text{for all} \quad l \geqslant 0, \quad m \geqslant k. \tag{3.1}$$

It is not true that only the independent schemes have the above property. The scheme $\{2, 2 : 2\}^2$ is a suitable counterexample.

Now, in view of Theorem 2.2, we have the following (cf. also Step 0)

Corollary 3.2. *Let* $\mathscr{N} = \{n_1, ..., n_s; n\}^k$ *be as in Theorem* 2.2; *then*

$$\sum_{i=1}^{s} \binom{n_i + m - 1}{m} \leqslant \binom{n+m}{m}, \qquad \text{for all} \quad m \geqslant k.$$

In particular, the following combinatorial inequality is an immediate consequence of Theorem 2.2.,

$$\sum_{i=1}^{s} \binom{n_i+k-1}{k} \leqslant \binom{n+k}{k}, \tag{3.2}$$

with $\mathcal{N}=\{n_1, ..., n_s; n\}^k$ as in Theorem 2.2.

The inequality (3.2) also can be proved directly, and for any sequence $\{n_1, ..., n_s, n\}$ of nonnegative numbers, by the use of the following theorem, which is a generalization of Lemma 2.1 from [1].

THEOREM 3.3A. *Let* $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_s$ *and* $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$ *be arbitrary collections of nonnegative numbers and* q, $1 \leqslant q \leqslant s$, *be an integer. Then*

$$\sum_{i=1}^{s} n_i m_i \leqslant \frac{M}{q} \sum_{i=1}^{q} n_i, \tag{3.3}$$

where

$$M=\max\left(qm_1, \sum_{i=1}^{s} m_i\right).$$

Moreover, the equality holds in (3.3) *if and only if*

(i) $qm_1 \leqslant \sum_{i=1}^{s} m_i$ *in the case of* $n_q \neq 0$,

and there is an integer j, $0 \leqslant j \leqslant q-1$, *for which*

(ii) $m_1 = \cdots = m_j = M/q$ *in the case of* $j \neq 0$,

(iii) $n_{j+1} = \cdots = n_l$ *with* $l=s$ *or* $q \leqslant l < s$ *and* $m_{l+1}=0$.

Proof. We have

$$\begin{aligned}
\sum_{i=1}^{s} n_i m_i &\leqslant \sum_{i=1}^{q-1} n_i m_i + n_q(m_q + \cdots + m_s) \\
&\leqslant \sum_{i=1}^{q-1} n_i m_i + n_q(M - m_1 - \cdots - m_{q-1}) \\
&\leqslant \sum_{i=1}^{q-1} n_i m_i + n_q(M - m_1 - \cdots - m_{q-1}) \\
&\quad + \sum_{i=1}^{q-1} (n_i - n_q)\left(\frac{M}{q} - m_i\right) \\
&= \frac{M}{q} \sum_{i=1}^{q} n_i,
\end{aligned}$$

thus establishing the inequality (3.3). Next we consider the case of equality in (3.3). The conditions (iii) and (i) above imply that we have equalities in the first and second places of the above chain of inequalities respectively. Also, the conditions (i) and (iii) ensure the equality in the third place.

On the other hand, if we have equality in the second place then (i) holds. The equality in the third place implies (ii) and $n_{j+1} = \cdots = n_q$. Finally, the equality in the first place extends the latter equalities up to the condition (iii). This completes the proof.

The following inequality is closely related to the above one. By means of these two inequalities one can estimate (above and below) any sum of form $\sum_{i=1}^{s} n_i m_i$, without any condition on the order of nonnegative numbers n_i and m_i.

THEOREM 3.3b. *Let* $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_s$ *and* $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$ *be arbitrary collections of nonnegative numbers and* q, $1 \leqslant q \leqslant s$, *be an integer. Then*

$$\sum_{i=1}^{s} n_i m_i \geqslant \frac{m}{q} \sum_{i=1}^{q} n_i, \tag{3.4}$$

where

$$m = \min\left(q m_1, \sum_{i=1}^{s} m_i\right).$$

Proof. To prove (3.4) we replace M by m and reverse inequality signs in the above chain of inequalities. Similarly, we obtain the following conditions for equality in (3.4):

(i) $q m_1 \geqslant \sum_{i=1}^{s} m_i$ in the case of $n_q \neq 0$,

and there is an integer j, $0 \leqslant j \leqslant q-1$, for which

(ii) $m_1 = \cdots = m_j = m/q$ in the case of $j \neq 0$,

(iii) $n_{j+1} = \cdots = n_l$ with $l = s$ or $q \leqslant l < s$ and $m_{l+1} = 0$.

Now let us prove the inequality (3.2) (the mentioned extension) directly by using Theorem 3.3a. Applying Theorem 3.3a for the sequences $\{n_i + 1\}_{i=1}^{s}$ and $\{n_i\}_{i=1}^{s}$, where the last one satisfies (2.2), we obtain

$$\sum_{i=1}^{s} n_i(n_i+1) \leqslant q^{k-2} A(A+q).$$

Then applying it again for the sequences $\{n_i + 2\}_{i=1}^{s}$ and $\{n_i(n_i+1)\}_{i=1}^{s}$, we obtain

$$\sum_{i=1}^{s} n_i(n_i+1)(n_i+2) \leqslant q^{k-3} A(A+q)(A+2q).$$

Continuing in this way we get

$$\sum_{i=1}^{s} n_i \cdots (n_i + k - 1) \leqslant A(A+q) \cdots (A + (k-1)\, q).$$

Finally, in view of the inequality

$$A(A+q) \cdots (A + (k-1)\, q) \leqslant (n+1) \cdots (n+k)$$

(which follows from the equality $A + (A + (k-1)\, q) = (n+1) + (n+k)$), we get (3.2).

Let us mention that similarly one can show that the inequalities (2.2) (for arbitrary A) imply

$$\sum_{i=1}^{s} (n_i)^j \leqslant q^{k-j} A^j, \qquad j = 1, ..., k.$$

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